

# LATTICE INVARIANTS FROM THE HEAT KERNEL

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**ABSTRACT.** We derive lattice invariants from the heat flux of a lattice. Using systems of harmonic polynomials, we obtain sums of products of spherical theta functions which give new invariants of integer lattices which are modular forms. In particular, we show that the modular forms  $\Theta_{nn,\Lambda}$  depend only from lengths and angles in the lattice.

## 1. INTRODUCTION

For a lattice  $\Lambda \subset \mathbb{E}^n$  one of the most important invariants is its theta function  $\Theta_\Lambda$ . This function does not determine the lattice. We want to describe and illustrate a way to derive more lattice invariants from the heat flux of the lattice. The first idea is to apply the heat flux to the lattice  $\Lambda$  considered as distribution on functions on  $\mathbb{E}^n$ . In doing so, we obtain a function  $f_t = f_{t,\Lambda} : \mathbb{R}^+ \times \mathbb{E}^n \rightarrow \mathbb{R}$ . Secondly, we try to deduce invariants from  $f_t$  which do not depend on the embedding of the lattice. The restriction of  $f$  to  $\mathbb{R}^+ \times \{0\}$  is a first such invariant. However it gives essentially the theta function of the lattice (see 2.10). An obvious way to obtain invariants is to integrate products homogeneous parts of  $f$  over the unit sphere  $S^{n-1} \subset \mathbb{E}^n$ . These invariants are denoted by  $c_{k_1, \dots, k_m}$  (see 2.7). In order to obtain modular forms, we study polynomial differentials of  $f$ , in particular for harmonic polynomials (see Proposition 2.5). Whenever we find an invariant sum of such products (which we call a harmonic datum), then we obtain an invariant modular form (see Proposition 2.9). Thus, we need a construction for harmonic data. We illustrate how the invariants  $c_{k_1, \dots, k_m}$  can be used to construct harmonic data in Section 3 where we construct the harmonic data  $p_{11}$  and  $p_{22}$  for lattices in  $\mathbb{E}^2$ . Seeing the results we obtain from these data (cf. Corollaries 3.3 and 3.5) it is not hard to show that for a lattice  $\Lambda \subset \mathbb{E}^2$  of level  $N$  the holomorphic function

$$\Theta_{nn,\Lambda}(\tau) = \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} (\cos(2n\angle(\gamma, \delta)) \|\gamma\|^{2n} \|\delta\|^{2n}) q^{\|\gamma\|^2 + \|\delta\|^2} \quad \text{with } q = \exp(2\pi i \tau)$$

is a modular form with integer coefficients of weight  $4n+2$  and of level  $N$  (see Theorem 3.6 for more details). In the final Section 4 we start to generalize the modular forms  $\Theta_{mm,\Lambda}$  to lattices in  $n$  dimensional euclidean space  $\mathbb{E}^n$ . In particular we deduce in Theorem 4.2 a concrete formula for  $\Theta_{11,\Lambda}$  which we apply to two isospectral lattices  $\Lambda_1$  and  $\Lambda_2$  in  $\mathbb{E}^4$  found by A. Schiemann in [4]. An elementary calculation (see Proposition 4.4) shows that  $\Theta_{11,\Lambda_1} \neq \Theta_{11,\Lambda_2}$ . Thus, the modular lattice invariant  $\Theta_{11,\Lambda}$  can distinguish at least two isospectral lattices.

*Notation:* By a lattice we mean a free  $\mathbb{Z}$  module  $\Lambda \cong \mathbb{Z}^n$  with a positive definite quadratic form  $q : \Lambda \rightarrow \mathbb{R}$ . A lattice possesses isometric embeddings  $\Lambda \rightarrow \mathbb{E}^n$ . Two such embeddings differ by an element of the orthogonal group  $O(n)$ . The number  $q(\gamma)$  is the length of  $\gamma \in \Lambda$ . The norm of  $\gamma \in \Lambda$  is defined to be  $\|\gamma\| := \sqrt{q(\gamma)}$ .

2. THE FUNCTION  $f_t$ 

**2.1. The heat flux of a lattice.** We consider a cocompact lattice  $\Lambda \subset \mathbb{E}^n$  in the  $n$  dimensional euclidean space. We consider two lattices  $\Lambda, \Lambda' \subset \mathbb{E}^n$  to be isomorphic when there exists an element  $\varphi \in O(n)$  which transforms  $\Lambda$  into  $\Lambda'$ , in short  $\Lambda' = \varphi(\Lambda)$ .

We regard  $\Lambda \in D^n(\mathbb{E}^n)$  as a distribution on the Schwartz space  $S(\mathbb{E}^n)$  by

$$\Lambda(f) = \sum_{\gamma \in \Lambda} f(\gamma).$$

Smooth  $n$  forms  $\omega$  on  $\mathbb{E}^n$  can be considered as currents  $[\omega] \in D^n(\mathbb{E}^n)$  by  $[\omega]f := \int_{\mathbb{E}^n} \omega \cdot f$ . A current  $[\omega]$  associated to a smooth form  $\omega$  is called a smooth current. We want to subdue the singular distribution  $\Lambda$  to the heat flux to obtain a smooth current  $(\Lambda)_t$  for any positive  $t \in \mathbb{R}$ . We follow the approach in [2, Section 2]:

$$\text{The Laplace operator is given by:} \quad \Delta = - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

For any  $k$ -form  $f \in A^k(\mathbb{E}^n)$  we have the heat flux of  $f$ :

$$P_t(f) \in A^k(X) \text{ for all } t \in \mathbb{R}_+ \quad \left( \frac{\partial}{\partial t} + \Delta \right) P_t(f) = 0 \quad \lim_{t \rightarrow 0} P_t(f) = f.$$

The heat flux of a distribution  $T \in D^n(\mathbb{E}^n)$  is defined by

$$P_t(T)(f) := T(P_t(f)).$$

If the distribution is smooth, then we have the equality:

$$P_t([\omega]) = [P_t(\omega)].$$

That is the extension to distributions is compatible with the embedding  $A^n(\mathbb{E}^n) \hookrightarrow D^n(\mathbb{E}^n)$  and the heat flux on smooth forms. Furthermore, for  $T = \Lambda$  the heat flux smoothes the distribution, that is

$$P_t(\Lambda) = [\omega_t] \quad \text{for some smooth } \omega_t \in A^n(\mathbb{E}^n).$$

Using the explicit form of the heat kernel for a point (cf. [1, Chapter 2]) we can write down the heat flux of  $\Lambda$  directly as  $\omega_t = f_t d\mu$ , with  $d\mu = dx_1 \wedge \dots \wedge dx_n$  the volume form on  $\mathbb{E}^n$ , and

$$f_t(x) = (4\pi t)^{-\frac{n}{2}} \sum_{\gamma \in \Lambda} \exp\left(\frac{-||x - \gamma||^2}{4t}\right).$$

**2.2.  $f_t$  determines the lattice**  $\Lambda \subset \mathbb{E}^n$ . Indeed we have on the level of distributions  $\lim_{t \rightarrow +0} [f_t d\mu] = \Lambda$ . However, the function  $f_t$  depends on the embedding of  $\Lambda$ . We try to deduce more informations by considering the values of derivatives of  $f_t$  along  $\mathbb{R}^+ \times \{0\}$ .

**2.3. Polynomial derivations.** We need the subring  $A = \mathbb{R}[x_1, \dots, x_n] \subset A^0(\mathbb{E}^n)$  of polynomials in the smooth functions. We define a pairing on

$$A \times A^0(\mathbb{E}^n) \rightarrow \mathbb{R} \quad \langle P, f \rangle = P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) f|_0.$$

Note that the pairing is also well defined when  $P$  is smooth and  $f$  is a polynomial. We have the following properties of that pairing:

- (i) The pairing is bilinear.
  - (ii) The pairing is symmetric, in the sense that  $\langle P, f \rangle = \langle f, P \rangle$  whenever one side is defined.
  - (iii) the restriction of  $\langle \cdot, \cdot \rangle$  to  $A \times A$  is positive definite.
  - (iv) The monomials form a orthogonal (not orthonormal!) basis.
  - (v) For two polynomials  $P, Q \in A$  we have  $\langle P \cdot Q, f \rangle = \langle P, Q \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) f \rangle$ .
- For any  $P \in A$  we obtain with  $\langle P, f_t \rangle$  a smooth function  $\langle P, f_t \rangle : \mathbb{R}^+ \rightarrow \mathbb{R}$ .

**Lemma 2.4.** *For the  $O(n)$ -invariant polynomial  $r^2 := \sum_{i=1}^n x_i^2$  we have*

$$\langle r^{2k}, f_t \rangle = \frac{\partial^k}{\partial t^k} f_t(0).$$

*Proof.* Having in mind, that  $f_t$  is the heat flux of  $\Lambda$ , that is  $(\Delta + \frac{\partial}{\partial t})f_t = 0$ , we get

$$\langle r^2, f_t \rangle = \langle 1, r^2 \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) f_t \rangle = \langle 1, -\Delta(f_t) \rangle = \langle 1, \frac{\partial}{\partial t} f_t \rangle = \frac{\partial}{\partial t} f_t(0).$$

Now the assertion of the lemma follows from 2.3 (v). □

So evaluation of  $O(n)$  invariant polynomials  $P = \sum_i a_i r^{2i}$  gives only expressions of type  $\sum_i a_i \frac{\partial^i}{\partial t^i} f_t(0)$ . After having seen what  $\langle r^{2k}, f_t \rangle$  is, we study the pairing  $\langle h, f_t \rangle$  for homogeneous harmonic polynomials. We have the following result:

**Proposition 2.5.** *If  $h \in A$  is an homogeneous harmonic polynomial of degree  $d$ , then*

$$(2t)^d \langle h, f_t \rangle = (4\pi t)^{\frac{-n}{2}} \sum_{\gamma \in \Lambda} h(\gamma) \exp\left(\frac{-\|\gamma\|^2}{4t}\right).$$

In order to prove this proposition we need an auxiliary result:

**Lemma 2.6.** *Let  $h$  be homogeneous polynomial of degree  $d$ ,  $a \in \mathbb{C}^*$  a number and  $f = f(x_1, \dots, x_n) = \exp\left(\frac{a}{2} \cdot \sum_{i=1}^n x_i^2\right)$ . Then we have an equality*

$$h \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_1} \right) f = a^d \left( \sum_{k \geq 0} \left( \frac{-1}{2a} \right)^k \frac{1}{k!} \Delta^k(h) \right) \cdot f.$$

*Proof.* We have to prove the lemma only for  $h$  a monomial. We proceed by induction on the degree of  $h$ . For  $\deg(h) = 0$ , the statement is obvious. Suppose the assertion holds for  $h'$  of degree  $d - 1$ . We set  $h = x_i h'$ . An direct calculation shows that

$$\Delta(h) = x_i \Delta(h') - 2 \frac{\partial}{\partial x_i} h'.$$

Since  $\Delta$  and  $\frac{\partial}{\partial x_i}$  commute we deduce from that formula inductively that

$$(1) \quad \Delta^k(h) = x_i \Delta^k(h') - 2k \frac{\partial}{\partial x_i} \Delta^{k-1}(h')$$

holds for all integers  $k \geq 0$ . Now we compute using the induction hypothesis, the Leibniz rule, and equation (1):

$$\begin{aligned} h \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_1} \right) f &= \frac{\partial}{\partial x_i} \left( a^{d-1} \left( \sum_{k \geq 0} \left( \frac{-1}{2a} \right)^k \frac{1}{k!} \Delta^k(h') \right) \cdot f \right) \\ &= a^d \left( \sum_{k \geq 0} \left( \frac{-1}{2a} \right)^k \frac{1}{k!} (x_i \Delta^k(h') - 2k \frac{\partial}{\partial x_i} \Delta^{k-1}(h')) \right) \cdot f \\ &= a^d \left( \sum_{k \geq 0} \left( \frac{-1}{2a} \right)^k \frac{1}{k!} \Delta^k(h) \right) \cdot f. \end{aligned} \quad \square$$

*Proof of Proposition 2.5.* Setting  $a = \frac{1}{2t}$  in Lemma 2.6 we obtain that

$$h\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_1}\right) \exp\left(\frac{-\sum_{i=1}^n x_i^2}{4t}\right) = \left(\frac{-1}{2t}\right)^d h(x_1, \dots, x_n) \exp\left(\frac{-\sum_{i=1}^n x_i^2}{4t}\right).$$

Setting  $x_i = \tilde{x}_i - \gamma_i$  for some  $\gamma = (\gamma_1, \dots, \gamma_n) \in \Lambda$  this reads

$$h\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_1}\right) \exp\left(\frac{-\|\gamma - x\|^2}{4t}\right) = \left(\frac{-1}{2t}\right)^d h(x_1 - \gamma_1, \dots, x_n - \gamma_n) \exp\left(\frac{-\|\gamma - x\|^2}{4t}\right)$$

Multiplying this equality with  $(4\pi t)^{\frac{-n}{2}}$  and summing up over all  $\gamma \in \Lambda$  yields the statement of the proposition when specializing to  $x = 0$ .  $\square$

**2.7. Definition of the  $c_{k_1, k_2, \dots, k_m}$ .** In order to find an invariant we first decompose the function  $f_t$  by in its homogeneous parts  $f_t = f_0 + f_1 + \dots$  with

$$f_k = \sum_{I \subset \mathbb{N}^n, |I|=2k} a_I \frac{x^I}{I!} \text{ where for } I = (i_1, i_2, \dots, i_n) \quad I! := \prod_{m=1}^n i_m! \text{ , } x^I := \prod_{m=1}^n x_m^{i_m} \text{ , and}$$

$$a_I := \langle x^I, f_t \rangle = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} f_t|_{x=0}$$

These polynomials are of course not invariant but the integrals of their products

$$c_{k_1, k_2, \dots, k_m} := \int_{S^{n-1}} f_{k_1} \cdot f_{k_2} \cdots f_{k_m} d\bar{\mu}$$

are. Here  $d\bar{\mu}$  denotes the normalized  $O(n)$  invariant measure on  $S^{n-1}$  such that  $\int_{S^{n-1}} d\bar{\mu} = 1$ . We restrict to the parts of even degree because the parts of odd degree vanish for  $f_t$  being an even function. We will see that the  $c_{k_1, k_2, \dots, k_m}$  can be expressed as products of expressions of type  $\langle h, f_t \rangle$ . The decomposition of a polynomial  $f \in A$  into a sum  $f = h_n + r^2 h_{n-2} + r^4 h_{n-4} + \dots$  with  $h_i$  harmonic will allow applications of the following:

**2.8. Spherical theta function principle.** We want to compute these invariants and relate them to modular forms for integral lattices. Here we call a lattice  $\Lambda \subset \mathbb{E}^n$  integral when  $\|\gamma\|^2 \in \mathbb{N}$  for all  $\gamma \in \Lambda$ . Such a lattice has a level  $N \in \mathbb{N}$  and a discriminant  $D$  (cf. [5, Chapter 3.1]). For an homogeneous harmonic function  $h$  on  $\mathbb{E}^n$  of degree  $d$  we have the spherical theta function  $\Theta_{h, \Lambda}$  defined by

$$\Theta_{h, \Lambda}(\tau) = \sum_{\gamma \in \Lambda} h(\gamma) q^{\|\gamma\|^2}, \quad q = \exp(2\pi i \tau).$$

These spherical theta functions are modular forms of weight  $\frac{n}{2} + d$ , level  $N$ , and character  $\left(\frac{D}{N}\right)$  (see again [5, Chapter 3.2]).

*Definition.* An harmonic datum  $p = ((h_{ij})_{i=1, \dots, m} j=1, \dots, k)$  for lattices in  $\mathbb{E}^n$  consists of harmonic homogeneous polynomials  $h_{ij}$  of degree  $d_i$  on  $\mathbb{E}^n$ , which define a map

$$p : \{\text{lattices } \Lambda \subset \mathbb{E}^n\} \rightarrow \{\text{smooth functions } \mathbb{R}^+ \rightarrow \mathbb{C}\} \quad \Lambda \mapsto p(\Lambda)$$

$$\text{by } p(\Lambda) := \sum_{j=1}^k \prod_{i=1}^m \langle h_{ij}, f_{t, \Lambda} \rangle \quad \text{where} \quad f_{t, \Lambda}(x) = (4\pi t)^{\frac{-n}{2}} \sum_{\gamma \in \Lambda} \exp\left(\frac{-\|x - \gamma\|^2}{4t}\right),$$

which satisfies  $p(\Lambda) = p(\varphi(\Lambda))$  for all  $\varphi \in O(n)$ . The sum  $d = \sum_{i=1}^m d_i$  is called the degree of the harmonic datum. The harmonic datum is called even (resp. odd) when  $m$  is even (resp. odd).

The connection between harmonic data and invariant modular forms is given by the following proposition.

**Proposition 2.9.** *Suppose we have a harmonic datum  $(p, (h_{ij})_{i=1,\dots,m} j=1,\dots,k)$  for lattices in  $\mathbb{E}^n$  of degree  $d$ . Then for any integral lattice  $\Lambda \subset \mathbb{E}^n$  of level  $N$  the modular form*

$$\Theta_{p,\Lambda}(\tau) := \sum_{j=1}^k \prod_{i=1}^m \Theta_{h_{ij},\Lambda}$$

*is invariant under  $O(n)$ .  $\Theta_{p,\Lambda}(\tau)$  is a modular form of weight  $\frac{nm}{2} + d$ , and of level  $N$ . If the harmonic system is odd, then  $\Theta_{p,\Lambda}(\tau)$  has character  $\left(\frac{D}{\cdot}\right)$  with  $D$  the discriminant of the lattice. If it is even, then  $\Theta_{p,\Lambda}(\tau)$  is a modular form for the trivial character.*

*Proof.* First we remark that  $\Theta_{p,\Lambda}$  being a sum of products of modular forms of level  $N$  is a modular form of level  $N$  and the stated weight. Likewise we see that it is a modular form of character  $\left(\frac{D}{\cdot}\right)^m$ . Thus, it remains to show that  $\Theta_{p,\Lambda}$  is invariant under  $O(n)$ .

We start with calculating  $p(\Lambda)(t)$  using Proposition 2.5:

$$p(\Lambda)(t) = \sum_{j=1}^k \prod_{i=1}^m \langle h_{ij}, f_{t,\Lambda} \rangle = \sum_{j=1}^k \prod_{i=1}^m (2t)^{-d_i} (4\pi t)^{-\frac{n}{2}} \sum_{\gamma \in \Lambda} h_{ij}(\gamma) \exp\left(\frac{-||\gamma||^2}{4t}\right)$$

Since the  $d_i$  sum up to  $d$  we get

$$(2t)^d (4\pi t)^{\frac{mn}{2}} p(\Lambda)(t) = \sum_{j=1}^k \prod_{i=1}^m \sum_{\gamma \in \Lambda} h_{ij}(\gamma) \exp\left(\frac{-||\gamma||^2}{4t}\right)$$

is a lattice invariant. However, this is the value of  $\Theta_{p,\Lambda}\left(\frac{i}{8\pi t}\right)$ , so by the identity theorem for holomorphic functions the values of the modular form  $\Theta_{p,\Lambda}$  along the imaginary axis determine this function.  $\square$

**2.10. Example: The theta series.** The function  $f_t(0) = (4\pi t)^{-\frac{n}{2}} \sum_{\gamma \in \Lambda} \exp(-||\gamma||^2/4t)$  is a lattice invariant. The theta function of the lattice  $\Lambda$  is

$$\Theta_{\Lambda}(\tau) := \sum_{\gamma \in \Lambda} \exp(2\pi i \tau ||\gamma||^2).$$

So up to the scaling factor  $(4\pi t)^{-\frac{n}{2}}$  the theta function  $\Theta_{\Lambda}\left(\frac{i}{8\pi t}\right)$  gives  $f_t(0)$ :

$$f_t(0) = (4\pi t)^{-\frac{n}{2}} \Theta_{\Lambda}\left(\frac{i}{8\pi t}\right).$$

Vice versa, from  $f_t(0)$  we can extract the values of  $\Theta_{\Lambda}$  along the imaginary line, which determines  $\Theta_{\Lambda}$  by the identity theorem for holomorphic functions.

Setting  $h_{11} \equiv 1$  we obtain a harmonic datum  $(f_t(0), 1)$  of degree zero with  $m = k = 1$ . This harmonic datum yields the theta function of the lattice up to the factor  $(4\pi t)^{\frac{n}{2}}$ . We show in the sequel that we can build more harmonic data from our lattice invariants  $c_{k_1, \dots, k_r}$ .

### 3. LATTICES IN $\mathbb{E}^2$

**3.1. Preparations.** Let  $\Lambda \subset \mathbb{E}^2$  be a fixed lattice. From Corollary A.2 we deduce the following table of integrals which will be useful for the coming calculations:

$$\begin{aligned}
\int_{S^1} x_0^2 d\bar{\mu} &= \frac{1}{2} & \int_{S^1} x_0^2 x_1^2 d\bar{\mu} &= \frac{1}{8} & \int_{S^1} x_0^4 d\bar{\mu} &= \frac{3}{8} \\
\int_{S^1} x_0^4 x_1^2 d\bar{\mu} &= \frac{1}{16} & \int_{S^1} x_0^6 d\bar{\mu} &= \frac{5}{16} & \int_{S^1} x_0^4 x_1^4 d\bar{\mu} &= \frac{3}{128} \\
\int_{S^1} x_0^6 x_1^2 d\bar{\mu} &= \frac{5}{128} & \int_{S^1} x_0^8 d\bar{\mu} &= \frac{35}{128}
\end{aligned}$$

By definition  $c_0 = f_t(0) = \frac{\Theta_\Lambda(\frac{i}{4\pi t})}{4\pi t}$ , so we see that  $c_0$  is essentially the theta function of the lattice. Next we compute

$$4c_1 = 4 \int_{S^1} \left( a_{20} \frac{x^2}{2} + a_{11} xy + a_{02} \frac{y^2}{2} \right) d\bar{\mu} = a_{20} + a_{02}.$$

Thus, we obtain  $c_1 = \frac{1}{4} \frac{\partial}{\partial t} c_0$ . This is a general pattern:  $c_2 = \frac{1}{64} \frac{\partial^2}{\partial t^2} c_0$ ,  $c_3 = \frac{1}{2304} \frac{\partial^3}{\partial t^3} c_0$ , and in general  $c_n = \frac{1}{4^n (n!)^2} \frac{\partial^n}{\partial t^n} c_0$ .

**3.2. The harmonic datum  $p_{11}$  for lattices in  $\mathbb{E}^2$ .** We start with the computation of the lattice invariant  $c_{11}$ :

$$\begin{aligned}
32c_{1,1} &= 32 \int_{S^1} \left( a_{20} \frac{x^2}{2} + a_{11} xy + a_{02} \frac{y^2}{2} \right)^2 d\bar{\mu} \\
&= 3a_{20}^2 + 3a_{02}^2 + 4a_{11}^2 + 2a_{20}a_{02} \\
&= 2(a_{20} + a_{02})^2 + 4a_{11}^2 + (a_{20} - a_{02})^2
\end{aligned}$$

The function  $p_{11} = 32c_{11} - 32c_1^2$  is obviously a lattice invariant, and gives a harmonic datum:

$$p_{11} = 4\langle x_0 x_1, f_t \rangle^2 + \langle x_0^2 - x_1^2, f_t \rangle^2$$

This harmonic datum yields by Proposition 2.9:

**Corollary 3.3.** *For any integral lattice  $\Lambda \subset \mathbb{E}^2$  of level  $N$  the modular form*

$$\Theta_{11,\Lambda} = 4\Theta_{x_0 x_1, \Lambda}^2 + \Theta_{x_0^2 - x_1^2, \Lambda}^2$$

*of weight 6 and level  $N$  is invariant under the  $O(2)$  action on the embedding  $\Lambda \rightarrow \mathbb{E}^2$ . If we write  $\Theta_{11,\Lambda}(\tau) = \sum_{n \in \mathbb{N}} a_n q^n$  with  $q = \exp(2\pi i \tau)$ , then the  $a_n$  are given by*

$$a_n = \sum_{\substack{(\gamma, \delta) \in \Lambda \times \Lambda \\ \|\gamma\|^2 + \|\delta\|^2 = n}} \cos(2\angle(\gamma, \delta)) \|\gamma\|^2 \|\delta\|^2.$$

*Proof.* The only thing which remains to be shown is the formula for the  $a_n$ . This is the result of a straightforward calculation, where we expand  $4\Theta_{x_0 x_1, \Lambda}^2$  and  $\Theta_{x_0^2 - x_1^2, \Lambda}^2$  as sums over  $\Lambda \times \Lambda$ , and check that the summand of  $q^{\|\gamma\|^2 + \|\delta\|^2}$  is  $\cos(2\angle(\gamma, \delta)) \|\gamma\|^2 \|\delta\|^2$  for each pair  $(\gamma, \delta) \in \Lambda \times \Lambda$ . See the proof of Theorem 3.6 for this calculation.  $\square$

**3.4. The harmonic datum  $p_{22}$  for lattices in  $\mathbb{E}^2$ .** We start with a decomposition of  $c_{22}$  into three summands

$$\begin{aligned}
73728c_{2,2} &= 6a_{40}a_{04} + 96a_{31}a_{13} + 60a_{22}a_{04} + 60a_{40}a_{22} + 108a_{22}^2 + \\
&\quad + 80a_{13}^2 + 35a_{40}^2 + 35a_{04}^2 + 80a_{31}^2 \\
&= p_{22} + s_1 + s_2 \\
\text{with } p_{22} &= (a_{40} - 6a_{22} + a_{04})^2 + 16(a_{31} - a_{13})^2 \\
s_1 &= 18(a_{40} + 2a_{22} + a_{04})^2 \\
s_2 &= 16((a_{20} + a_{02}) \circ (a_{20} - a_{02}))^2 + 64((a_{20} + a_{02}) \circ a_{11})^2
\end{aligned}$$

If we can show that  $s_1$  and  $s_2$  are invariant under the  $O(2)$  action, then it follows that

$p_{22}$  is also invariant under that action. However, we see that  $s_1 = 73728c_2^2$ , which is an invariant. Next we see that

$$s_2 = 16 \left( 4 \langle x_0 x_1, \frac{\partial}{\partial t} f_t \rangle^2 + \langle x_0^2 - x_1^2, \frac{\partial}{\partial t} f_t \rangle^2 \right).$$

As we have seen before Corollary 3.3 this is also invariant under the  $O(2)$  action. Eventually we come up with a further harmonic datum:

$$p_{22} = \langle x_0^4 - 6x_0^2 x_1^2 + x_1^4, f_t \rangle^2 + 16 \langle x_0 x_1 (x_0^2 - x_1^2), f_t \rangle^2.$$

Using this harmonic datum gives by Proposition 2.9:

**Corollary 3.5.** *For a integral lattice  $\Lambda \subset \mathbb{E}^2$  the modular form*

$$\Theta_{22,\Lambda} = \Theta_{x_0^4 - 6x_0^2 x_1^2 + x_1^4, \Lambda}^2 + 16 \Theta_{x_0 x_1 (x_0^2 - x_1^2), \Lambda}^2$$

*of weight 10 is invariant under the  $O(2)$  action on the embedding  $\Lambda \rightarrow \mathbb{E}^2$ . The  $q$ -expansion of  $\Theta_{22,\Lambda} = \sum_{n \geq 1} a_n q^n$  is given by*

$$a_n = \sum_{\substack{(\gamma, \delta) \in \Lambda \times \Lambda \\ \|\gamma\|^2 + \|\delta\|^2 = n}} \cos(4\angle(\gamma, \delta)) \|\gamma\|^4 \|\delta\|^4.$$

*Proof.* It is enough to derive from the expression  $\Theta_{22,\Lambda} = \Theta_{x_0^4 - 6x_0^2 x_1^2 + x_1^4, \Lambda}^2 + 16 \Theta_{x_0 x_1 (x_0^2 - x_1^2), \Lambda}^2$  the stated  $q$ -expansion which is an elementary calculation. However, since this is a special case of Theorem 3.6, we omit the proof.  $\square$

**Theorem 3.6.** *For a positive integer  $n \in \mathbb{Z}$  we take the two harmonic polynomials  $h_1(x, y) = \operatorname{Re}((x + iy)^{2n})$ , and  $h_2(x, y) = \operatorname{Im}((x + iy)^{2n})$  of degree  $2n$ . For any integral lattice  $\Lambda \subset \mathbb{E}^2$  of level  $N$  there is a  $O(2)$ -invariant modular form*

$$\Theta_{nn,\Lambda}(\tau) := \Theta_{h_1,\Lambda}^2(\tau) + \Theta_{h_2,\Lambda}^2(\tau)$$

*of level  $N$ , and weight  $2 + 4n$ . Its  $q$ -expansion  $\Theta_{nn,\Lambda}(\tau) = \sum_{n \geq 0} a_n q^n$  is given by*

$$a_n = \sum_{\substack{(\gamma, \delta) \in \Lambda \times \Lambda \\ \|\gamma\|^2 + \|\delta\|^2 = n}} \cos(2n\angle(\gamma, \delta)) \|\gamma\|^{2n} \|\delta\|^{2n}.$$

*Furthermore,  $\Theta_{nn,\Lambda} \in (4q^{2k}) \subset \mathbb{Z}[[q]]$  where  $k$  is the first minimal length of the lattice.*

*Proof.* We compute the  $q$ -expansion of  $\Theta_{nn,\Lambda}$ .

$$\begin{aligned} \Theta_{nn,\Lambda}(\tau) &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} (h_1(\gamma)h_1(\delta) + h_2(\gamma)h_2(\delta)) q^{\|\gamma\|^2 + \|\delta\|^2} \\ &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} (\operatorname{Re}(\gamma^{2n}) \operatorname{Re}(\delta^{2n}) + \operatorname{Im}(\gamma^{2n}) \operatorname{Im}(\delta^{2n})) q^{\|\gamma\|^2 + \|\delta\|^2} \\ &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} (\operatorname{Re}(\gamma^{2n} \bar{\delta}^{2n})) q^{\|\gamma\|^2 + \|\delta\|^2} \\ &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} \left( \operatorname{Re} \left( \frac{\gamma^{2n} \bar{\delta}^{2n}}{\gamma^n \bar{\delta}^n} \right) \gamma^n \bar{\gamma}^n \delta^n \bar{\delta}^n \right) q^{\|\gamma\|^2 + \|\delta\|^2} \\ &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} (\cos(2n\angle(\gamma, \delta)) \|\gamma\|^{2n} \|\delta\|^{2n}) q^{\|\gamma\|^2 + \|\delta\|^2} \end{aligned}$$

We deduce from this form of the  $q$ -expansion that  $a_0 = a_1 = \dots = a_{2k-1} = 0$ , and that  $\Theta_{nn,\Lambda}$  does not depend from the embedding  $\Lambda \subset \mathbb{E}^2$ . It remains to demonstrate that  $a_n \in 4\mathbb{Z}$ .

To show this, we use from the above deduction that  $a_n = \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} \operatorname{Re}(\gamma^{2n} \bar{\delta}^{2n})$  where we sum over all pairs  $(\gamma, \delta)$  with  $\|\gamma\|^2 + \|\delta\|^2 = n$ . The symmetric bilinear form  $\psi : \Lambda \times \Lambda \rightarrow \mathbb{R}$  with  $\psi(\gamma, \delta) := \operatorname{Re}(\gamma \bar{\delta})$  satisfies  $\psi(\gamma, \gamma) \in \mathbb{Z}$  because  $\Lambda$  is integral. Thus,  $\psi(\gamma, \delta) \in \frac{1}{2}\mathbb{Z}$ . We

deduce that for  $\gamma, \delta \in \Lambda$  the element  $x := \gamma\bar{\delta}$  satisfies a quadratic equation  $x^2 - b_1x + b_2 = 0$  with  $b_1 = 2\operatorname{Re}(\gamma\bar{\delta}) \in \mathbb{Z}$ , and  $b_2 = \|\gamma\|^2\|\delta\|^2 \in \mathbb{Z}$ . Thus,  $x$  is an integer in a (at most) quadratic field extension  $K/\mathbb{Q}$ . Consequently,  $x^{2n}$  is an integer in this field  $K$  which implies  $2\operatorname{Re}(\gamma^{2n}\bar{\delta}^{2n}) \in \mathbb{Z}$ . If  $\gamma \neq \pm\delta$ , then the summand  $\operatorname{Re}(\gamma^{2n}\bar{\delta}^{2n})$  appears eight times in  $a_n$ , namely from the pairs  $(\pm\gamma, \pm\delta)$  and  $(\pm\delta, \pm\gamma)$ . If  $\gamma = \pm\delta$ , then  $\operatorname{Re}(\gamma^{2n}\bar{\delta}^{2n})$  is an integer and appears four times.  $\square$

*Example 1. Some invariants for the lattice  $\Lambda$  associated to  $q(x, y) = x^2 + y^2$ .*

$$\begin{aligned}\Theta_\Lambda(\tau) &= 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + 8q^{13} + 4q^{16} + 8q^{17} + \dots \\ \Theta_{22,\Lambda}(\tau) &= 16(q^2 - 8q^3 + 16q^4 + 32q^5 - 156q^6 + 112q^7 + 256q^8 - 576q^9 + \dots) \\ \Theta_{44,\Lambda}(\tau) &= 16(q^2 + 32q^3 + 256q^4 + 512q^5 + 6084q^6 - 33728q^7 + 65536q^8 + \dots) \\ \Theta_{11,\Lambda}(\tau) &= 0 = \Theta_{33,\Lambda}(\tau)\end{aligned}$$

*Example 2. Some invariants for the lattice  $\Lambda$  associated to  $q(x, y) = x^2 + xy + y^2$ .*

$$\begin{aligned}\Theta_\Lambda(\tau) &= 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + 12q^{19} + \dots \\ \Theta_{33,\Lambda}(\tau) &= 36(q^2 - 54q^4 + 128q^5 + 729q^6 - 3456q^7 + 3524q^8 + 16902q^{10} + \dots) \\ \Theta_{11,\Lambda}(\tau) &= 0 = \Theta_{22,\Lambda}(\tau) = \Theta_{44,\Lambda}(\tau)\end{aligned}$$

#### 4. THE HARMONIC DATUM $p_{11}$ FOR LATTICES IN $\mathbb{E}^n$

**4.1. An explicit formula for  $c_{11}$  and definition of  $p_{11}$ .** Let  $\Lambda \subset \mathbb{E}^n$  be a lattice. As explained in 2.1 the heat flux of the distribution  $\Lambda$  is given by  $\omega_{\Lambda,t} = f_t dx_1 \wedge \dots \wedge dx_n$  with

$$f_t(x) = (4\pi t)^{-\frac{n}{2}} \sum_{\gamma \in \Lambda} \exp\left(\frac{-\|x - \gamma\|^2}{4t}\right).$$

The value of  $f_t(0)$  is the first lattice invariant. To derive the lattice invariant  $c_{11}$  we need the second Taylor coefficient of  $f_t$  in zero. We write it in the following manner:

$$f_{t,1} = \sum_{i=1}^n \frac{a_i}{2} x_i^2 + \sum_{1 \leq i < j \leq n} b_{ij} x_i x_j$$

where  $a_i = \frac{\partial^2}{\partial x_i^2} f_t|_{x=0}$  and  $b_{ij} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f_t|_{x=0}$ . The invariant  $c_{11}$  is defined to be the integral  $c_{11} := \int_{S^{n-1}} f_{t,1}^2 d\bar{\mu}$ .

As a preparation we compute  $c_1 := \int_{S^{n-1}} f_{t,1} d\bar{\mu}$ . Using the integral formulas  $\int_{S^{n-1}} x_i^2 d\bar{\mu} = \frac{1}{n}$  and  $\int_{S^{n-1}} x_i x_j d\bar{\mu} = 0$  (cf. Corollary A.2) we see that

$$c_1 = \frac{1}{2n} \left( \sum_{i=1}^n a_i \right) = \frac{-1}{2n} \Delta f_t|_{x=0} = \frac{1}{2n} \frac{\partial f_t}{\partial t} \Big|_{x=0}.$$

We need the following integrals from Corollary A.2:

$$\int_{S^{n-1}} x_i^4 d\bar{\mu} = \frac{3}{n(n+2)}, \quad \int_{S^{n-1}} x_i^2 x_j^2 d\bar{\mu} = \frac{1}{n(n+2)}.$$

Having in mind (Corollary A.2) that the integrals of homogeneous monomials over  $S^{n-1}$  vanish if an odd exponent occurs we find the following:

$$\begin{aligned}4n(n+2)c_{11} &= 4 \sum_{1 \leq i < j \leq n} b_{ij}^2 + 3 \sum_{i=1}^n a_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j \\ &= 4 \sum_{1 \leq i < j \leq n} b_{ij}^2 + 2 \sum_{i=1}^n a_i^2 + \sum_{i=1}^n \sum_{j=1}^n a_i a_j.\end{aligned}$$



We want to express  $c_{11}$  in terms of differential operators which correspond either to harmonic polynomials or to  $r := a_1 + a_2 + \dots + a_n$ . This  $r$  corresponds to  $-\Delta$ , as  $r = -\Delta(f)|_{x=0}$ . So we introduce the harmonic polynomials  $h_i = na_i - r$ . Using the obvious equality  $\sum_{i=1}^n h_i = 0$  we obtain:

$$\begin{aligned} 4n^3(n+2)c_{11} &= 4n^2 \sum_{1 \leq i < j \leq n} b_{ij}^2 + 2 \sum_{i=1}^n (na_i)^2 + \sum_{i=1}^n \sum_{j=1}^n (na_i)(na_j) \\ &= 4n^2 \sum_{1 \leq i < j \leq n} b_{ij}^2 + 2 \sum_{i=1}^n (h_i + r)^2 + \sum_{i=1}^n \sum_{j=1}^n (h_i + r)(h_j + r) \\ &= \left( 4n^2 \sum_{1 \leq i < j \leq n} b_{ij}^2 + 2 \sum_{i=1}^n h_i^2 \right) + ((n^2 + 2n)r^2). \end{aligned}$$

So the invariant  $c_{11}$  decomposes as  $c_{11} = \frac{1}{2n^3(n+2)}p_{11} + \frac{r^2}{4n^2}$ . The second summand corresponds to  $c_1^2 = \left(\frac{1}{2n}\frac{\partial}{\partial r}f_t|_{x=0}\right)^2$ . Hence, the first summand is also invariant. It yields a harmonic datum:

$$p_{11} = 2n^2 \sum_{1 \leq i < j \leq n} b_{ij}^2 + \sum_{i=1}^n h_i^2 = 2n^2 \sum_{1 \leq i < j \leq n} \langle x_i x_j, f_t \rangle^2 + \sum_{i=1}^n \langle h_i, f_t \rangle^2.$$

*Remark.* Note that for  $n = 2$  we have  $h_1 = -h_2$ , so  $h_1^2 = h_2^2$ . Thus, in this case we have  $p_{11} = 2 \left( 4 \left( \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} f_t|_{x=0} \right)^2 + \left( \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) f_t|_{x=0} \right)^2 \right)$ . Up to the factor 2 this is our harmonic system from 3.2.

Applying Proposition 2.9 we obtain the next result:

**Theorem 4.2.** *Let  $\Lambda \subset \mathbb{E}^n$  be an integral lattice, of level  $N$ . The modular form*

$$\Theta_{11,\Lambda}(\tau) = 2n^2 \left( \sum_{1 \leq i < j \leq n} \Theta_{x_i x_j, \Lambda}^2(\tau) \right) + \sum_{i=1}^n \Theta_{nx_i^2 - \sum_{j=1}^n x_j^2, \Lambda}^2(\tau)$$

*is independent from the embedding  $\Lambda \rightarrow \mathbb{E}^n$ .  $\Theta_{11,\Lambda}$  is a cusp form of weight  $n+4$  of level  $N$ . Its  $q$ -expansion is given by*

$$\Theta_{11,\Lambda}(\tau) = \sum_{m \geq 0} a_m q^m \quad \text{with} \quad a_m = n^2 \sum_{\substack{(\gamma, \delta) \in \Lambda \times \Lambda \\ \|\gamma\|^2 + \|\delta\|^2 = m}} \left( \cos^2(\angle(\gamma, \delta)) - \frac{1}{n} \right) \|\gamma\|^2 \|\delta\|^2.$$

*We have  $\Theta_{11,\Lambda} \in (2nq^{2k}) \subset \mathbb{Z}[[q]]$  with  $k$  the first minimal length of the lattice  $\Lambda$ . If  $n$  is even, then  $\Theta_{11,\Lambda} \in (4nq^{2k}) \subset \mathbb{Z}[[q]]$ .*

*Proof.* On the one hand, when considering the  $q$ -expansion it is obvious that  $\Theta_{11,\Lambda}$  is invariant under the  $O(n)$  action. On the other hand, being a sum of squares of modular forms of weight  $\frac{n}{2} + 2$  forces  $\Theta_{11,\Lambda}$  to be a modular form of weight  $n+4$ . Thus, even though  $\Theta_{11,\Lambda}$  comes from a harmonic datum, all we have to do is to show that the modular form has the stated  $q$ -expansion.

$$\begin{aligned} \Theta_{11,\Lambda}(\tau) &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} \left( 2n^2 \sum_{1 \leq i < j \leq n} \gamma_i \gamma_j \delta_i \delta_j + \sum_{i=1}^n (n\gamma_i^2 - \|\gamma\|^2)(n\delta_i^2 - \|\delta\|^2) \right) q^{\|\gamma\|^2 + \|\delta\|^2} \\ &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} \left( n^2 \sum_{i=1}^n \sum_{j=1}^n \gamma_i \gamma_j \delta_i \delta_j - n\|\gamma\|^2 \|\delta\|^2 \right) q^{\|\gamma\|^2 + \|\delta\|^2} \\ &= \sum_{(\gamma, \delta) \in \Lambda \times \Lambda} (n^2 \langle \gamma, \delta \rangle^2 - n\|\gamma\|^2 \|\delta\|^2) q^{\|\gamma\|^2 + \|\delta\|^2} \end{aligned}$$

Now the definition of the cosine gives the formula for the  $q$ -expansion. From this formula we conclude that  $a_0 = a_1 = \dots = a_{2k-1} = 0$ . In order to prove that  $a_m \in 2n\mathbb{Z}$  we consider the sum

$$\frac{1}{n}a_m = \sum_{\substack{(\gamma, \delta) \in \Lambda \times \Lambda \\ \|\gamma\|^2 + \|\delta\|^2 = m}} (n\langle \gamma, \delta \rangle^2 - \|\gamma\|^2 \|\delta\|^2)$$

Since  $\Lambda$  is integral we have  $n\langle \gamma, \delta \rangle^2 \in \frac{1}{4}\mathbb{Z}$  (respectively in  $\frac{1}{2}\mathbb{Z}$  when  $n$  is even). If  $\gamma \neq \pm\delta$ , then the eight pairs  $(\pm\gamma, \pm\delta)$  and  $(\pm\delta, \pm\gamma)$  in  $\Lambda \times \Lambda$  give the same contribution  $(n\langle \gamma, \delta \rangle^2 - \|\gamma\|^2 \|\delta\|^2)$  to  $a_m$ . If  $\gamma \neq \pm\delta$ , then  $\langle \gamma, \delta \rangle$  is an integer and the integer summand  $(n\langle \gamma, \delta \rangle^2 - \|\gamma\|^2 \|\delta\|^2)$  appears four times.  $\square$

#### 4.3. Example: Computing $\Theta_{11, \Lambda}$ for two isospectral lattices in dimension four.

We consider the two integral lattices  $\Lambda_1$  and  $\Lambda_2$  in  $\mathbb{E}^4$  which were investigated by A. Schiemann in [4]. The Gram matrices of these lattices are given by

$$A_1 = \frac{1}{2} \begin{pmatrix} 4 & 2 & 0 & 1 \\ 2 & 8 & 3 & 1 \\ 0 & 3 & 10 & 5 \\ 1 & 1 & 5 & 10 \end{pmatrix} \quad A_2 = \frac{1}{2} \begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 8 & 1 & -4 \\ 1 & 1 & 8 & 2 \\ 1 & -4 & 2 & 10 \end{pmatrix}$$

Schiemann showed in his article that these two lattices are not isometric even though they give the same theta function  $\Theta(\tau) = \Theta_{\Lambda_1}(\tau) = \Theta_{\Lambda_2}(\tau)$  which he determined to be

$$\Theta(\tau) = 1 + 2q^2 + 4q^4 + 6q^5 + 10q^6 + 6q^7 + 12q^8 + 6q^9 + 6q^{10} + 8q^{11} + 10q^{12} + 8q^{13} + 10q^{14} + 22q^{15} + \dots$$

**Proposition 4.4.** *We have an inequality  $\Theta_{11, \Lambda_1} \neq \Theta_{11, \Lambda_2}$ . Thus, the modular forms  $\Theta_{11, \Lambda_1}$  and  $\Theta_{11, \Lambda_2}$  of level 1729 and weight eight distinguish the two isospectral lattices  $\Lambda_1$  and  $\Lambda_2$ .*

*Proof.* We need embeddings of  $\Lambda_i \rightarrow \mathbb{E}^4$ . We choose a decomposition  $A_i = S_i^t \cdot S_i$  with  $S_i$  upper triangular. We obtain

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 1 & 0 & \frac{1}{2} \\ 0 & \sqrt{7} & \frac{3\sqrt{7}}{7} & \frac{\sqrt{7}}{14} \\ 0 & 0 & \frac{\sqrt{427}}{7} & \frac{67\sqrt{427}}{854} \\ 0 & 0 & 0 & \frac{\sqrt{105469}}{122} \end{pmatrix}$$

The column vectors  $\{\gamma_i\}_{i=1,2,3,4}$  of the matrix  $S_1$  generate the lattice  $\Lambda_1$ . From the theta function we see that there are two vectors of norm  $\sqrt{2}$  ( $\pm\gamma_1$ ), and four vectors of norm two ( $\pm\gamma_2$ , and  $\pm(\gamma_1 - \gamma_2)$ ). Fortunately, these six lattice vectors are enough. We compute now in the ring  $A = \mathbb{R}[[q]]$ . The six spherical theta functions  $\Theta_{\Lambda_1, x_i x_j}$  for  $i < j$  have zero coefficient at  $q^2$ . Since there are no lattice vectors of norm  $\sqrt{3}$  we find  $\Theta_{\Lambda_1, x_i x_j} \in (q^4)$ . Thus, we deduce  $\Theta_{\Lambda_1, x_i x_j}^2 \in (q^8)$ .

Considering the lattice vectors of norm at most 2, we see

$$\begin{aligned} \Theta_{\Lambda_1, 4x_1^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= 12q^2 - 8q^4 + \dots \\ \Theta_{\Lambda_1, 4x_2^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= -4q^2 + 40q^4 + \dots \\ \Theta_{\Lambda_1, 4x_3^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= -4q^2 - 16q^4 + \dots \\ \Theta_{\Lambda_1, 4x_4^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= -4q^2 - 16q^4 + \dots \end{aligned}$$

Taking the squares of these equations we obtain

$$\begin{aligned}\Theta_{\Lambda_1, 4x_1^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}^2 &= 144q^4 - 192q^6 + \dots \\ \Theta_{\Lambda_1, 4x_2^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}^2 &= 16q^4 - 320q^6 + \dots \\ \Theta_{\Lambda_1, 4x_3^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}^2 &= 16q^4 + 128q^6 + \dots \\ \Theta_{\Lambda_1, 4x_4^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}^2 &= 16q^4 + 128q^6 + \dots\end{aligned}$$

Summing up these four squares we obtain that

$$\Theta_{11, \Lambda_1}(\tau) = 192q^4 - 256q^6 + \dots$$

We repeat this construction now with the lattice  $\Lambda_2$ . In this case we find

$$S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2\sqrt{2} & \frac{\sqrt{2}}{4} & -\sqrt{2} \\ 0 & 0 & \frac{\sqrt{122}}{4} & \frac{9\sqrt{122}}{122} \\ 0 & 0 & 0 & \frac{\sqrt{105469}}{122} \end{pmatrix}$$

The vectors of norm  $\sqrt{2}$  are  $\pm\gamma_1$ , whereas the vectors of norm two are the four vectors  $\pm\gamma_2$ , and  $\pm\gamma_3$ . Again, we have  $\Theta_{\Lambda_2, x_i x_j}^2 \in (q^8)$  for  $i < j$ . As before we compute

$$\begin{aligned}\Theta_{\Lambda_2, 4x_1^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= 12q^2 - 15q^4 + \dots \\ \Theta_{\Lambda_2, 4x_2^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= -4q^2 + \frac{33}{2}q^4 + \dots \\ \Theta_{\Lambda_2, 4x_3^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= -4q^2 + \frac{29}{2}q^4 + \dots \\ \Theta_{\Lambda_2, 4x_4^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)} &= -4q^2 - 16q^4 + \dots\end{aligned}$$

Eventually, we obtain  $\Theta_{11, \Lambda_2}(\tau) = 192q^4 - 480q^6 + \dots \neq \Theta_{11, \Lambda_1}(\tau)$ .  $\square$

*Remark.* To see that the two modular forms are different it is enough to compute up to the first Fourier coefficient which is different. Anyway, using a computer we can give more coefficients:

$$\begin{aligned}\Theta_{11, \Lambda_1}(\tau) &= 192q^4 - 256q^6 - 896q^7 + 1120q^8 - 2848q^9 + 3024q^{10} - 2112q^{11} + \\ &\quad + 13536q^{12} - 4064q^{13} - 16272q^{14} - 4544q^{15} + \dots \\ \Theta_{11, \Lambda_2}(\tau) &= 192q^4 - 480q^6 - 608q^7 + 736q^8 - 1312q^9 + 3216q^{10} + 1056q^{11} - \\ &\quad - 2048q^{12} - 2624q^{13} + 2896q^{14} - 12288q^{15} + \dots\end{aligned}$$

## APPENDIX A. INTEGRATING POLYNOMIALS ON SPHERES

Let  $f \in \mathbb{R}[x_0, x_1, \dots, x_n]$  be a polynomial. We need the integral  $\int_{S^n} f d\mu$  for further computations. Here  $d\mu$  denotes the standard  $O(n+1)$  invariant measure on  $S^n$ . Since we easily can decompose  $f$  into its homogeneous components, it is enough to consider homogeneous polynomials  $f$ . Here we have the following result:

**Proposition A.1.** *Let  $f \in \mathbb{R}[x_0, x_1, \dots, x_n]$  be an homogeneous polynomial of degree  $d$ . If  $d$  is odd, we have that  $\int_{S^n} f d\mu = 0$ . If  $d = 2k$  is even,  $\Delta^k f$  is a real number, and we have*

$$\int_{S^n} f d\mu = c_d \Delta^k f \quad \text{with} \quad c_d := \alpha_d \cdot \frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})}, \quad \text{and} \quad \alpha_{2k} := \frac{1}{(-2)^k k! \prod_{m=1}^k (n+2m-1)}.$$

*Proof.* We start with the following observation (see Exercise 33 on page 550 in [3]):  $f$  can be uniquely decomposed

$$f = \sum_{l=0}^{\lfloor \frac{d}{2} \rfloor} r^l h_{d-2l}$$

where  $r = \sum_{i=0}^n x_i^2$  and the  $h_i$  are homogeneous harmonic polynomials of degree  $i$ , that is  $\Delta h_i = 0$ . The mean value principle and the fact that  $r$  is constantly one on  $S^n$  together with the fact that the volume of  $S^n$  is  $\frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})}$  yield

$$\int_{S^n} f = \int_{S^n} \sum_{l=0}^{d/2} h_{d-2l} = \frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})} \sum_{l=0}^{d/2} h_{d-2l}(0)$$

Now the value of an homogeneous polynomial at zero is zero, unless the degree of the polynomial is zero. Thus, we deduce that  $\int_{S^n} f d\mu = \frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})} h_0$ . This, shows that integration of homogeneous polynomials of odd degree gives zero. Next, we show (by simple and straightforward differentiation) the following equality for any homogeneous harmonic polynomial  $h$ :

$$\Delta(r^m h) = (-2m)(n + 2m + 2 \deg(h) - 1)r^{m-1}h.$$

In consequence we have  $\Delta^k r^k = (-2)^k k! \prod_{m=1}^k (n + 2m - 1) = \frac{1}{\alpha_{2k}}$ , and  $\Delta^k r^m h = 0$  for  $m < k$  and  $h$  harmonic. We deduce that  $\Delta^k f$  yields  $\frac{h_0}{\alpha_{2k}}$ , which finishes the proof.  $\square$

We need concrete formulas for the normalized measure  $d\bar{\mu} = \frac{\Gamma(\frac{n+3}{2})}{(n+1)\pi^{\frac{n+1}{2}}} d\mu$ . This measure is  $O(n)$ -invariant and has the property that  $\int_{S^n} d\bar{\mu} = 1$ . As an application of the above Proposition A.1 we obtain:

**Corollary A.2.** *For the monomial  $f = x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$  of degree  $d = i_0 + i_1 + \cdots + i_n$  we have the equality*

$$\int_{S^n} f d\bar{\mu} = \begin{cases} 0 & \text{at least one of the } i_k \text{ is odd.} \\ \left( \prod_{k=0}^n \frac{i_k!}{2^{\frac{i_k}{2}}} \right) \left( \prod_{m=1}^{d/2} \frac{1}{2(n + 2m - 1)} \right) & \text{all the } i_k \text{ are even.} \end{cases}$$

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